**Physics I**
**ISI B.Math**
**Final : November 23, 2022**

Total Marks: 50
Time : 3 hours

Answer all questions:

1. (**Marks** $= \mathbf{5} \times \mathbf{2}$)

Choose the correct option

i) Which of the following statements is not true in general about a closed system of $N$ particles whose total linear momentum is conserved.

a) The Lagrangian of the system is invariant under spatial translations of the entire system .
b) The mutual internal force between any two particles of the system must act along the straight line joining the two particles.
c) The centre of mass of the system moves with uniform velocity.
d) The motion of the centre of mass is independent of the specific nature of the internal forces.

ii) The number of degrees of freedom of four rigid rods flexibly jointed to form a quadrilateral which can slide on a flat table is

a) 6
b) 2
c) 4
d) 3

iii) A particle of mass $m$ is moving under the influence of a force $\mathbf{F} = -k\mathbf{r}$. Which of the following statements is false ?

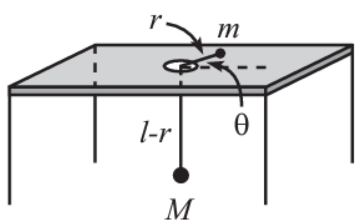
a) Angular momentum about the origin is conserved.
b) Total energy is conserved
c) The orbit of the particle is a hyperbola
d) The orbit of the particle is an ellipse

iv) Which of the following Lagrangians does not correctly describe the one dimensional motion of a mass $m$ attached to a spring of spring constant $k$ ?

a) $L = \frac{1}{2}m\dot{x}^2 - \sqrt{mk}x\dot{x} - \frac{1}{2}kx^2$
b) $L = \frac{1}{2}m\dot{x}^2 + \alpha\dot{x} - \frac{1}{2}kx^2$ (where $\alpha$ is a constant)
c) $L = \frac{1}{2}m\dot{x}^2 + \beta x\dot{x}^2 - \frac{1}{2}kx^2$ (where $\beta$ is a constant)
d) $L = \frac{1}{2}m\dot{x}^2 + \gamma x^2\dot{x} - \frac{1}{2}kx^2$ (where $\gamma$ is a constant)

v) A uniform ball of mass $M$ and radius $a$ is pivoted so that it can turn freely about one of its diameters which is fixed in a vertical position. A beetle of mass $m$ can crawl on the surface of the ball. Initially the ball is rotating with angular speed $\Omega$ with the beetle at the North pole. The beetle walks to the equator and stops. The final angular velocity $\Omega'$ of the ball is

(a) equal to $\Omega$
(b) greater than $\Omega$
(c) less than $\Omega$
(d) less than or greater than $\Omega$ depending on the path taken by the beetle to reach the equator.




2. (**Marks = 3 + 2 + 2 + 3**)

A particle P of mass $m$ is free to slide on a frictionless table and is connected via a string of length $l$ that passes through a hole O in the table to a mass $M$ that hangs below. Assume that $M$ moves in vertical line only. Suppose that the length of the part of the string on the table is $r(t)$ at a time $t$ and $\theta(t)$ is the angle between OP and some fixed reference line on the table that passes through O .

(a) Write down the Lagrangian of the system in terms of the generalized coordinates $(r, \theta)$ and hence find the equations of motion for the system.

(b) Identify the cyclic coordinate and find the corresponding conserved generalized momentum. What is the symmetry of the Lagrangian that is associated with this conserved quantity ?

(c) If the mass $m$ undergoes circular motion, find the radius $r_0$ of the circular orbit in terms of the conserved generalized momentum.

(d) Show that the angular frequency of small radial oscillations about this circular orbit is $\omega = \sqrt{\frac{3M}{M+m}}\sqrt{\frac{g}{r_0}}$.

3. (**Marks = 5 + 3 + 2**)

(a) Find the inertia tensor of a homogeneous cube of density $\rho$, mass $M$ and side $a$ when one corner is at the origin and the three adjacent edges lie along the positive $x_1$, $x_2$,$x_3$ coordinate axes. Recall that the form of the inertia tensor is given by $I_{ij} = \int_V \rho(\mathbf{r}) \left(\delta_{ij} \sum_k {x_k}^2 - x_i x_j\right) dv$ where $dv = dx_1 dx_2 dx_3$ is the element of volume at the position defined by the vector $\mathbf{r}$, and where $V$ is the volume of the body.

(b) In the example of part (b), do the principal axes of inertia lie along the coordinate axes ? Explain. If the above cube rotates about the $x_3$ axis with an angular velocity $\omega = \omega\mathbf{e_3}$, where $\mathbf{e_3}$ is a unit vector in the $x_3$ direction, find the component $L_3$ of the angular momentum along the same direction. Will the angular momentum vector point in the same direction ?

(c) If one has a square of uniform density with one corner at the origin,and the two adjacent sides starting from the origin lying along the $x$ and $y$ axes respectively, find the principal axes for the square using symmetry arguments.

4. **(Marks: 4 + 3 + 3 )**

(a) Two particles with mass $m_1$ and $m_2$ interact with gravitational forces( $\mathbf{F} = -\frac{Gm_1m_2}{r^2}\hat{\mathbf{r}}$) They start out from rest a distance $\rho$ apart and are allowed to fall into each other. How long does it take for them to collide ?
(You may need to use the following result : $\int_0^1 \frac{dx}{\sqrt{\frac{1}{x}-1}} = \frac{\pi}{2}$)

(b) A particle is moving under the influence of an attractive central force given by $\mathbf{F}(\mathbf{r}) = -k\mathbf{r}$. Show that the radius vector $\mathbf{r}$ sweeps out equal areas in equal times.

(c) Assume Earth's orbit to be circular and that the Sun's mass suddenly decreases by half. What orbit does the Earth then have ? Will Earth escape the solar system ? Justify your answer.

5.(**Marks** $= \mathbf{5} \times \mathbf{2}$)

State whether the following statements are true or false with a very brief (one or two lines) explanation.
i) A particle moving under the potential $U(r) = Kr^5$ where $K$ is a positive constant can have a circular orbit
ii) A point mass $m$ travels in a circle of radius $R$ with centre at the point $(0, 0, z_0)$ with the plane of the circle parallel to the $x-y$ plane with an angular velocity $\omega\hat{\mathbf{z}}$.The angular momentum vector about the origin points in the same direction as the angular velocity.
iii) The point of support of a simple pendulum oscillates according to the equation $x = a\cos\omega t$. The total energy of the pendulum is not conserved.
iv) Newton's Third Law does not hold for the case of a binary star system of two stars in orbit around each other under the influence of their mutual gravitational interaction.
v) The angular acceleration $\dot{\omega}$ of a body measured in a fixed inertial system is identical to that measured with respect to a rotating coordinate system fixed in the body